Volumen de una esfera de radio $R$:

$$V=\frac{4}{3}\pi R^3$$

Demostración:

Sea $S=\left\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2+z^2=R^2\right\}$. Se tiene que

$$\lambda(S)=\int_S d(x,y,z)=\int_A \rho^2\operatorname{sen}\varphi d(\rho,\theta,\varphi)=\int_0^R\left[\int_0^{2\pi}\left(\int_0^{\pi}\rho^2\operatorname{sen}\varphi d\varphi\right)d\theta\right]d\rho=\int_0^R\left[\int_0^{2\pi}2\rho^2 d\theta\right]d\rho=$$

$$=\int_0^R 4\pi\rho^2 d\rho=\frac{4}{3}\pi R^3$$

donde $A=\left\{(\rho,\theta,\varphi): 0\le\rho\le R,\ 0\le\theta<2\pi,\ 0\le\varphi<\pi\right\}$ y $\lambda$ es la medida de Lebesgue.

C.Q.D.

Demostración:

$$V=\int_{-R}^{R}\pi\left[f(x)\right]^2 dx=\pi\int_{-R}^{R}\left(\sqrt{R^2-x^2}\right)^2 dx=2\pi\int_0^R\left(R^2-x^2\right)dx=2\pi\left(R^2x-\frac{x^3}{3}\right)\Bigg|_0^R=\frac{4}{3}\pi R^3$$

C.Q.D.

Área de una esfera de radio $R$:

$$A=4\pi R^2$$

Demostración:

$$A_{\text{esfera}}=2A_{\text{semiesfera}}=2\cdot 2\pi\int_0^R\sqrt{R^2-x^2}\sqrt{1+\frac{x^2}{R^2-x^2}}dx=4\pi R\int_0^R\sqrt{R^2-x^2}\,\frac{1}{\sqrt{R^2-x^2}}dx=4\pi R\int_0^R 1dx=$$

$$=4\pi R\,x\Big|_0^R=4\pi R^2$$

C.Q.D.