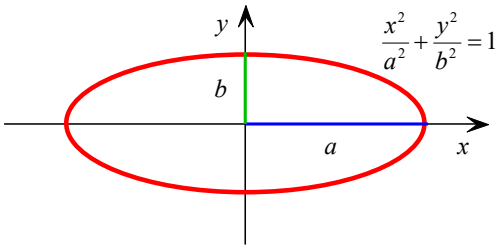

Longitud de la elipse

Sea $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ la elipse. Entonces

$$L=4\int_0^a\sqrt{1+\frac{a^2x^2}{a^2\left(a^2-x^2\right)}}dx=\frac{4}{a}\int_0^a\sqrt{\frac{a^4-a^2x^2+b^2x^2}{a^2\left(a^2-x^2\right)}}dx=\left[\begin{matrix}x=a\,\text{sen}\,\theta\\dx=a\cos\theta d\theta\end{matrix}\right]=$$

$$=\frac{4}{a}\int_0^{\pi/2}\sqrt{a^4-a^2\left(a^2-b^2\right)\text{sen}^2\,\theta}d\theta=\left[e=\frac{\sqrt{a^2-b^2}}{a}=\frac{c}{a}\right]=$$

$$=4a\int_0^{\pi/2}\sqrt{1-e^2\,\text{sen}^2\,\theta}d\theta\ \left(\begin{matrix}\text{Integral elíptica}\\\text{completa de 2ª especie}\end{matrix}\right)=$$

$$=2a\pi\left(1-\sum_{n=1}^{+\infty}\frac{\left[(2n)!\right]^2}{\left(2^n\cdot n!\right)^4}\cdot\frac{e^{2n}}{2n-1}\right),\ \text{donde}\ e=\frac{\sqrt{a^2-b^2}}{a}=\frac{c}{a}\ \text{es la excentricidad.}$$

Aproximaciones:

(1) Fórmula de Ramanujan

$$L\simeq\pi\left[3(a+b)-\sqrt{(3a+b)(a+3b)}\right]$$

(2) Fórmula de Ramanujan II

$$L\simeq\pi(a+b)\left[1+\frac{3\left(\frac{a-b}{a+b}\right)^2}{10+\sqrt{4-3\left(\frac{a-b}{a+b}\right)}}\right]$$

(3) Fórmula de Ramanujan II-Cantrel: $H=\left(\frac{a-b}{a+b}\right)^2$

$$L\simeq\pi(a+b)\left[1+\frac{3H}{10\sqrt{4-3H}}+\left(\frac{4}{\pi}-\frac{14}{11}\right)H^{12}\right]$$

(4) Fórmula de Gauss-Krammer

$$L\simeq\pi(a+b)\left[1+\left(\frac{1}{2}\right)^2H-\left(\frac{1}{2\cdot4}\right)^2H^2+\left(\frac{1}{2\cdot4\cdot6}\right)^2H^3+\ldots\right]=\pi(a+b)\sum_{n=0}^{+\infty}\binom{0,5}{n}^2H^n$$

Área de la elipse

Las ecuaciones paramétricas de la elipse son: $\begin{cases}x=a\cos t\\y=b\,\text{sen}\,t\end{cases}$, donde $x\in[-a,a]$ y, como consecuencia, $t\in[\pi,0]$. Así, el área de la elipse es:

$$A = 2\int_{\pi}^{0} (b\,\mathrm{sen}\,t)(-a\,\mathrm{sen}\,t)\,dt = -2ab\int_{\pi}^{0} \mathrm{sen}^2\,t\,dt = 2ab\int_{0}^{\pi} \mathrm{sen}^2\,t\,dt = 2ab\int_{0}^{\pi} \frac{1-\cos 2t}{2}\,dt =$$

$$= 2ab\left[\frac{t}{2} - \frac{\mathrm{sen}\,2t}{4}\right]\Bigg|_{0}^{\pi} = 2ab\frac{\pi}{2} = \pi ab$$

Dicha fórmula se puede obtener también mediante integración doble:

$$E_1 = \left\{(x,y) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+ : \frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1\right\}$$

$$A_1 = \int_{E_1} 1d(x,y) = \iint_{E_1} 1dxdy = \int_0^a \left(\int_0^{\frac{b}{a}\sqrt{a^2-x^2}} dy \right) dx = \int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx = \frac{b}{a}\int_0^a \sqrt{a^2-x^2}\,dx =$$

$$= \left[\begin{array}{lll} x = a\,\mathrm{sen}\,t & \Rightarrow & dx = a\cos t dt \\ x = 0 & \Rightarrow & t = 0 \\ x = a & \Rightarrow & t = \dfrac{\pi}{2} \end{array}\right] = \frac{b}{a}\int_0^a \sqrt{a^2 - a^2\,\mathrm{sen}^2\,t}\cdot a\cos t dt = b\int_0^a \sqrt{1-\mathrm{sen}^2\,t}\,\cos t dt =$$

$$= ab\int_0^a \cos^2 t dt = ab\int_0^a \frac{1+\cos 2t}{2}\,dt = ab\left[\frac{1}{2} + \frac{\mathrm{sen}\,2t}{4}\right]\Bigg|_0^a = \frac{\pi}{4}ab$$

$$\Rightarrow A_{\text{Elipse}} = 4A_1 = 4\frac{\pi}{4}ab = \pi ab$$

C.Q.D.