

La longitud de una circunferencia de radio r viene dada por:

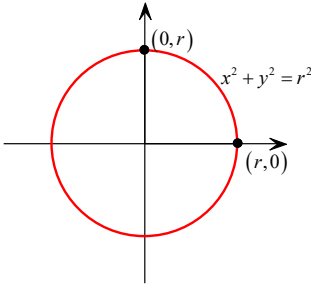
$$L = 2\pi r$$

Demostración:

La longitud de una curva $y = f(x)$ entre dos valores a y b viene dada por:

$$L = \int_a^b \sqrt{1 + [f'(x)]^2} dx$$

En nuestro caso, se tiene:



$$y^2 = r^2 - x^2 \Rightarrow y = \sqrt{r^2 - x^2}$$

$$f(x) = \sqrt{r^2 - x^2}$$

$$f'(x) = \frac{-2x}{2\sqrt{r^2 - x^2}} = \frac{-x}{\sqrt{r^2 - x^2}}$$

y, por tanto,

$$L = 4 \int_0^r \sqrt{1 + \frac{x^2}{r^2 - x^2}} dx = 4 \int_0^r \sqrt{\frac{r^2}{r^2 - x^2}} dx = 4 \int_0^r \frac{r}{\sqrt{r^2 - x^2}} dx = \left[\begin{array}{l} \text{sen } \theta = \frac{x}{r} \Rightarrow x = r \text{ sen } \theta \\ dx = r \text{ cos } \theta d\theta \\ x = 0 \Rightarrow \theta = 0 \\ x = r \Rightarrow \theta = \frac{\pi}{2} \end{array} \right] =$$

$$= 4 \int_0^{\pi/2} \frac{r^2 \text{ cos } \theta d\theta}{\sqrt{r^2 (1 - \text{sen}^2 \theta)}} = 4 \int_0^{\pi/2} \frac{r^2 \text{ cos } \theta d\theta}{\sqrt{r^2} \sqrt{1 - \text{sen}^2 \theta}} = 4 \int_0^{\pi/2} \frac{r^2 \text{ cos } \theta d\theta}{r \text{ cos } \theta} = 4r \int_0^{\pi/2} d\theta = 4r \frac{\pi}{2} = 2\pi r$$

C.Q.D.

El área de un círculo de radio r viene dada por:

$$A = \pi r^2$$

Demostración:

$$A = 2 \int_{-r}^r \sqrt{r^2 - x^2} dx = 4 \int_0^r \sqrt{r^2 - x^2} dx = \left[\begin{array}{l} x = r \text{ sen } t \Rightarrow dx = r \text{ cos } t dt \\ x = 0 \Rightarrow t = 0 \\ x = r \Rightarrow t = \frac{\pi}{2} \end{array} \right] = 4 \int_0^{\pi/2} \sqrt{r^2 - r^2 \text{ sen}^2 t} \cdot r \text{ cos } t dt =$$

$$= 4 \int_0^{\pi/2} r^2 \text{ cos}^2 t dt = 4r^2 \int_0^{\pi/2} \frac{1 + \text{cos } 2t}{2} dt = 4r^2 \left(\frac{1}{2} t + \frac{\text{sen } 2t}{4} \right) \Big|_0^{\pi/2} = 4r^2$$

C.Q.D.