# First International Olympiad, 1959

## 1959/1.

Prove that the fraction $\frac{21n+4}{14n+3}$ is irreducible for every natural number $n$.

## 1959/2.

For what real values of $x$ is

$$\sqrt{(x+\sqrt{2x-1})}+\sqrt{(x-\sqrt{2x-1})}=A,$$

given (a) $A=\sqrt{2}$, (b) $A=1$, (c) $A=2$, where only non-negative real numbers are admitted for square roots?

## 1959/3.

Let $a, b, c$ be real numbers. Consider the quadratic equation in $\cos x$ :

$$a\cos^2 x+b\cos x+c=0.$$

Using the numbers $a, b, c$, form a quadratic equation in $\cos 2x$, whose roots are the same as those of the original equation. Compare the equations in $\cos x$ and $\cos 2x$ for $a=4, b=2, c=-1$.

## 1959/4.

Construct a right triangle with given hypotenuse $c$ such that the median drawn to the hypotenuse is the geometric mean of the two legs of the triangle.

## 1959/5.

An arbitrary point $M$ is selected in the interior of the segment $AB$. The squares $AMCD$ and $MBEF$ are constructed on the same side of $AB$, with the segments $AM$ and $MB$ as their respective bases. The circles circumscribed about these squares, with centers $P$ and $Q$, intersect at $M$ and also at another point $N$. Let $N'$ denote the point of intersection of the straight lines $AF$ and $BC$.
(a) Prove that the points $N$ and $N'$ coincide.
(b) Prove that the straight lines $MN$ pass through a fixed point $S$ independent of the choice of $M$.
(c) Find the locus of the midpoints of the segments $PQ$ as $M$ varies between $A$ and $B$.

**1959/6.**

Two planes, $P$ and $Q$, intersect along the line $p$. The point $A$ is given in the plane $P$, and the point $C$ in the plane $Q$; neither of these points lies on the straight line $p$. Construct an isosceles trapezoid $ABCD$ (with $AB$ parallel to $CD$) in which a circle can be inscribed, and with vertices $B$ and $D$ lying in the planes $P$ and $Q$ respectively.

# Second International Olympiad, 1960

## 1960/1.

Determine all three-digit numbers $N$ having the property that $N$ is divisible by 11, and $N/11$ is equal to the sum of the squares of the digits of $N$.

## 1960/2.

For what values of the variable $x$ does the following inequality hold:

$$\frac{4x^2}{(1-\sqrt{1+2x})^2} < 2x+9?$$

## 1960/3.

In a given right triangle $ABC$, the hypotenuse $BC$, of length $a$, is divided into $n$ equal parts ($n$ an odd integer). Let $\alpha$ be the acute angle subtending, from $A$, that segment which contains the midpoint of the hypotenuse. Let $h$ be the length of the altitude to the hypotenuse of the triangle. Prove:

$$\tan\alpha = \frac{4nh}{(n^2-1)a}.$$

## 1960/4.

Construct triangle $ABC$, given $h_a, h_b$ (the altitudes from $A$ and $B$) and $m_a$, the median from vertex $A$.

## 1960/5.

Consider the cube $ABCDA'B'C'D'$ (with face $ABCD$ directly above face $A'B'C'D'$).
(a) Find the locus of the midpoints of segments $XY$, where $X$ is any point of $AC$ and $Y$ is any point of $B'D'$.
(b) Find the locus of points $Z$ which lie on the segments $XY$ of part (a) with $ZY = 2XZ$.

## 1960/6.

Consider a cone of revolution with an inscribed sphere tangent to the base of the cone. A cylinder is circumscribed about this sphere so that one of its bases lies in the base of the cone. Let $V_1$ be the volume of the cone and $V_2$ the volume of the cylinder.
(a) Prove that $V_1 \neq V_2$.
(b) Find the smallest number $k$ for which $V_1 = kV_2$, for this case, construct the angle subtended by a diameter of the base of the cone at the vertex of the cone.

**1960/7.**

An isosceles trapezoid with bases $a$ and $c$ and altitude $h$ is given.

(a) On the axis of symmetry of this trapezoid, find all points $P$ such that both legs of the trapezoid subtend right angles at $P$.

(b) Calculate the distance of $P$ from either base.

(c) Determine under what conditions such points $P$ actually exist. (Discuss various cases that might arise.)

# Third International Olympiad, 1961

## 1961/1.

Solve the system of equations:

$$\begin{aligned} x + y + z &= a \\ x^2 + y^2 + z^2 &= b^2 \\ xy &= z^2 \end{aligned}$$

where $a$ and $b$ are constants. Give the conditions that $a$ and $b$ must satisfy so that $x, y, z$ (the solutions of the system) are distinct positive numbers.

## 1961/2.

Let $a, b, c$ be the sides of a triangle, and $T$ its area. Prove: $a^2+b^2+c^2 \geq 4\sqrt{3}T$. In what case does equality hold?

## 1961/3.

Solve the equation $\cos^n x - \sin^n x = 1$, where $n$ is a natural number.

## 1961/4.

Consider triangle $P_1P_2P_3$ and a point $P$ within the triangle. Lines $P_1P$, $P_2P$, $P_3P$ intersect the opposite sides in points $Q_1, Q_2, Q_3$ respectively. Prove that, of the numbers

$$\frac{P_1P}{PQ_1}, \frac{P_2P}{PQ_2}, \frac{P_3P}{PQ_3}$$

at least one is $\leq 2$ and at least one is $\geq 2$.

## 1961/5.

Construct triangle $ABC$ if $AC = b, AB = c$ and $\angle AMB = \omega$, where $M$ is the midpoint of segment $BC$ and $\omega < 90°$. Prove that a solution exists if and only if

$$b \tan \frac{\omega}{2} \leq c < b.$$

In what case does the equality hold?

**1961/6.**

Consider a plane $\varepsilon$ and three non-collinear points $A, B, C$ on the same side of $\varepsilon$; suppose the plane determined by these three points is not parallel to $\varepsilon$. In plane a take three arbitrary points $A', B', C'$. Let $L, M, N$ be the midpoints of segments $AA', BB', CC'$; let $G$ be the centroid of triangle $LMN$. (We will not consider positions of the points $A', B', C'$ such that the points $L, M, N$ do not form a triangle.) What is the locus of point $G$ as $A', B', C'$ range independently over the plane $\varepsilon$?

## Fourth International Olympiad, 1962

### 1962/1.

Find the smallest natural number $n$ which has the following properties:
(a) Its decimal representation has 6 as the last digit.
(b) If the last digit 6 is erased and placed in front of the remaining digits, the resulting number is four times as large as the original number $n$.

### 1962/2.

Determine all real numbers $x$ which satisfy the inequality:

$$\sqrt{3-x} - \sqrt{x+1} > \frac{1}{2}.$$

### 1962/3.

Consider the cube $ABCDA'B'C'D'$ ($ABCD$ and $A'B'C'D'$ are the upper and lower bases, respectively, and edges $AA', BB', CC', DD'$ are parallel). The point $X$ moves at constant speed along the perimeter of the square $ABCD$ in the direction $ABCDA$, and the point $Y$ moves at the same rate along the perimeter of the square $B'C'CB$ in the direction $B'C'CBB'$. Points $X$ and $Y$ begin their motion at the same instant from the starting positions $A$ and $B'$, respectively. Determine and draw the locus of the midpoints of the segments $XY$.

### 1962/4.

Solve the equation $\cos^2 x + \cos^2 2x + \cos^2 3x = 1$.

### 1962/5.

On the circle $K$ there are given three distinct points $A, B, C$. Construct (using only straightedge and compasses) a fourth point $D$ on $K$ such that a circle can be inscribed in the quadrilateral thus obtained.

### 1962/6.

Consider an isosceles triangle. Let $r$ be the radius of its circumscribed circle and $\rho$ the radius of its inscribed circle. Prove that the distance d between the centers of these two circles is

$$d = \sqrt{r(r-2\rho)}.$$

### 1962/7.

The tetrahedron $SABC$ has the following property: there exist five spheres, each tangent to the edges $SA, SB, SC, BCCA, AB$, or to their extensions.
(a) Prove that the tetrahedron $SABC$ is regular.
(b) Prove conversely that for every regular tetrahedron five such spheres exist.

# Sixth International Olympiad, 1964

## 1964/1.

(a) Find all positive integers $n$ for which $2^n - 1$ is divisible by 7.
(b) Prove that there is no positive integer $n$ for which $2^n + 1$ is divisible by 7.

## 1964/2.

Suppose $a, b, c$ are the sides of a triangle. Prove that

$$a^2(b + c - a) + b^2(c + a - b) + c^2(a + b - c) \leq 3abc.$$

## 1964/3.

A circle is inscribed in triangle $ABC$ with sides $a, b, c$. Tangents to the circle parallel to the sides of the triangle are constructed. Each of these tangents cuts off a triangle from $\Delta ABC$. In each of these triangles, a circle is inscribed. Find the sum of the areas of all four inscribed circles (in terms of $a, b, c$).

## 1964/4.

Seventeen people correspond by mail with one another - each one with all the rest. In their letters only three different topics are discussed. Each pair of correspondents deals with only one of these topics. Prove that there are at least three people who write to each other about the same topic.

## 1964/5.

Suppose five points in a plane are situated so that no two of the straight lines joining them are parallel, perpendicular, or coincident. From each point perpendiculars are drawn to all the lines joining the other four points. Determine the maximum number of intersections that these perpendiculars can have.

## 1964/6.

In tetrahedron $ABCD$, vertex $D$ is connected with $D_0$ the centroid of $\Delta ABC$. Lines parallel to $DD_0$ are drawn through $A, B$ and $C$. These lines intersect the planes $BCD, CAD$ and $ABD$ in points $A_1, B_1$ and $C_1$, respectively. Prove that the volume of $ABCD$ is one third the volume of $A_1B_1C_1D_0$. Is the result true if point $D_0$ is selected anywhere within $\Delta ABC$?

# Seventh Internatioaal Olympiad, 1965

## 1965/1.

Determine all values $x$ in the interval $0 \leq x \leq 2\pi$ which satisfy the inequality

$$2\cos x \leq \left|\sqrt{1+\sin 2x} - \sqrt{1-\sin 2x}\right| \leq \sqrt{2}.$$

## 1965/2.

Consider the system of equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= 0 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= 0 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= 0 \end{aligned}$$

with unknowns $x_1, x_2, x_3$. The coefficients satisfy the conditions:
(a) $a_{11}, a_{22}, a_{33}$ are positive numbers;
(b) the remaining coefficients are negative numbers;
(c) in each equation, the sum of the coefficients is positive.
Prove that the given system has only the solution $x_1 = x_2 = x_3 = 0$.

## 1965/3.

Given the tetrahedron $ABCD$ whose edges $AB$ and $CD$ have lengths $a$ and $b$ respectively. The distance between the skew lines $AB$ and $CD$ is $d$, and the angle between them is $\omega$. Tetrahedron $ABCD$ is divided into two solids by plane $\varepsilon$, parallel to lines $AB$ and $CD$. The ratio of the distances of $\varepsilon$ from $AB$ and $CD$ is equal to $k$. Compute the ratio of the volumes of the two solids obtained.

## 1965/4.

Find all sets of four real numbers $x_1, x_2, x_3, x_4$ such that the sum of any one and the product of the other three is equal to 2.

## 1965/5.

Consider $\Delta OAB$ with acute angle $AOB$. Through a point $M \neq O$ perpendiculars are drawn to $OA$ and $OB$, the feet of which are $P$ and $Q$ respectively. The point of intersection of the altitudes of $\Delta OPQ$ is $H$. What is the locus of $H$ if $M$ is permitted to range over (a) the side $AB$, (b) the interior of $\Delta OAB$?

**1965/6.**

In a plane a set of $n$ points ($n \geq 3$) is given. Each pair of points is connected by a segment. Let $d$ be the length of the longest of these segments. We define a diameter of the set to be any connecting segment of length $d$. Prove that the number of diameters of the given set is at most $n$.

# Eighth International Olympiad, 1966

## 1966/1.

In a mathematical contest, three problems, $A, B, C$ were posed. Among the participants there were 25 students who solved at least one problem each. Of all the contestants who did not solve problem $A$, the number who solved $B$ was twice the number who solved $C$. The number of students who solved only problem $A$ was one more than the number of students who solved $A$ and at least one other problem. Of all students who solved just one problem, half did not solve problem $A$. How many students solved only problem $B$?

## 1966/2.

Let $a, b, c$ be the lengths of the sides of a triangle, and $\alpha, \beta, \gamma$, respectively, the angles opposite these sides. Prove that if

$$a + b = \tan\frac{\gamma}{2}(a\tan\alpha + b\tan\beta),$$

the triangle is isosceles.

## 1966/3.

Prove: The sum of the distances of the vertices of a regular tetrahedron from the center of its circumscribed sphere is less than the sum of the distances of these vertices from any other point in space.

## 1966/4.

Prove that for every natural number $n$, and for every real number $x \neq k\pi/2^t (t = 0, 1, ..., n; k$ any integer)

$$\frac{1}{\sin 2x} + \frac{1}{\sin 4x} + \cdots + \frac{1}{\sin 2^n x} = \cot x - \cot 2^n x.$$

## 1966/5.

Solve the system of equations

$$\begin{array}{llllll} & |a_1 - a_2|\, x_2 & + |a_1 - a_3|\, x_3 & + |a_1 - a_4|\, x_4 & = 1 \\ |a_2 - a_1|\, x_1 & & + |a_2 - a_3|\, x_3 & + |a_2 - a_3|\, x_3 & = 1 \\ |a_3 - a_1|\, x_1 & + |a_3 - a_2|\, x_2 & & & = 1 \\ |a_4 - a_1|\, x_1 & + |a_4 - a_2|\, x_2 & + |a_4 - a_3|\, x_3 & & = 1 \end{array}$$

where $a_1, a_2, a_3, a_4$ are four different real numbers.

**1966/6.**

In the interior of sides $BC, CA, AB$ of triangle $ABC$, any points $K, L, M$, respectively, are selected. Prove that the area of at least one of the triangles $AML, BKM, CLK$ is less than or equal to one quarter of the area of triangle $ABC$.

# Ninth International Olympiad, 1967

## 1967/1.

Let $ABCD$ be a parallelogram with side lengths $AB = a, AD = 1$, and with $\angle BAD = \alpha$. If $\Delta ABD$ is acute, prove that the four circles of radius 1 with centers $A, B, C, D$ cover the parallelogram if and only if

$$a \leq \cos\alpha + \sqrt{3}\sin\alpha.$$

## 1967/2.

Prove that if one and only one edge of a tetrahedron is greater than 1, then its volume is $\leq 1/8$.

## 1967/3.

Let $k, m, n$ be natural numbers such that $m + k + 1$ is a prime greater than $n + 1$. Let $c_s = s(s + 1)$. Prove that the product

$$(c_{m+1} - c_k)(c_{m+2} - c_k)\cdots(c_{m+n} - c_k)$$

is divisible by the product $c_1c_2\cdots c_n$.

## 1967/4.

Let $A_0B_0C_0$ and $A_1B_1C_1$ be any two acute-angled triangles. Consider all triangles $ABC$ that are similar to $\Delta A_1B_1C_1$ (so that vertices $A_1, B_1, C_1$ correspond to vertices $A, B, C$, respectively) and circumscribed about triangle $A_0B_0C_0$ (where $A_0$ lies on $BC, B_0$ on $CA$, and $AC_0$ on $AB$). Of all such possible triangles, determine the one with maximum area, and construct it.

## 1967/5.

Consider the sequence $\{c_n\}$, where

$$\begin{array}{rcl} c_1 & = & a_1 + a_2 + \cdots + a_8 \\ c_2 & = & a_1^2 + a_2^2 + \cdots + a_8^2 \\ & & \cdots \\ c_n & = & a_1^n + a_2^n + \cdots + a_8^n \\ & & \cdots \end{array}$$

in which $a_1, a_2, \cdots, a_8$ are real numbers not all equal to zero. Suppose that an infinite number of terms of the sequence $\{c_n\}$ are equal to zero. Find all natural numbers $n$ for which $c_n = 0$.

**1967/6.**

In a sports contest, there were $m$ medals awarded on $n$ successive days ($n > 1$). On the first day, one medal and 1/7 of the remaining $m - 1$ medals were awarded. On the second day, two medals and 1/7 of the now remaining medals were awarded; and so on. On the $n$-th and last day, the remaining $n$ medals were awarded. How many days did the contest last, and how many medals were awarded altogether?

# Tenth International Olympiad, 1968

## 1968/1.

Prove that there is one and only one triangle whose side lengths are consecutive integers, and one of whose angles is twice as large as another.

## 1968/2.

Find all natural numbers $x$ such that the product of their digits (in decimal notation) is equal to $x^2 - 10x - 22$.

## 1968/3.

Consider the system of equations

$$\begin{aligned} ax_1^2 + bx_1 + c &= x_2 \\ ax_2^2 + bx_2 + c &= x_3 \\ &\cdots \\ ax_{n-1}^2 + bx_{n-1} + c &= x_n \\ ax_n^2 + bx_n + c &= x_1, \end{aligned}$$

with unknowns $x_1, x_2, \cdots, x_n$, where $a, b, c$ are real and $a \neq 0$. Let $\Delta = (b-1)^2 - 4ac$. Prove that for this system
(a) if $\Delta < 0$, there is no solution,
(b) if $\Delta = 0$, there is exactly one solution,
(c) if $\Delta > 0$, there is more than one solution.

## 1968/4.

Prove that in every tetrahedron there is a vertex such that the three edges meeting there have lengths which are the sides of a triangle.

## 1968/5.

Let $f$ be a real-valued function defined for all real numbers $x$ such that, for some positive constant $a$, the equation

$$f(x+a) = \frac{1}{2} + \sqrt{f(x) - [f(x)]^2}$$

holds for all $x$.
(a) Prove that the function $f$ is periodic (i.e., there exists a positive number $b$ such that $f(x+b) = f(x)$ for all $x$).
(b) For $a = 1$, give an example of a non-constant function with the required properties.

**1968/6.**

For every natural number $n$, evaluate the sum

$$\sum_{k=0}^{\infty} \left[\frac{n+2^k}{2^{k+1}}\right] = \left[\frac{n+1}{2}\right] + \left[\frac{n+2}{4}\right] + \cdots + \left[\frac{n+2^k}{2^{k+1}}\right] + \cdots$$

(The symbol $[x]$ denotes the greatest integer not exceeding $x$.)

# Eleventh International Olympiad, 1969

## 1969/1.

Prove that there are infinitely many natural numbers $a$ with the following property: the number $z = n^4 + a$ is not prime for any natura1 number $n$.

## 1969/2.

Let $a_1, a_2, \cdots, a_n$ be real constants, $x$ a real variable, and

$$\begin{aligned} f(x) &= \cos(a_1 + x) + \frac{1}{2}\cos(a_2 + x) + \frac{1}{4}\cos(a_3 + x) \\ &\quad + \cdots + \frac{1}{2^{n-1}}\cos(a_n + x). \end{aligned}$$

Given that $f(x_1) = f(x_2) = 0$, prove that $x_2 - x_1 = m\pi$ for some integer $m$.

## 1969/3.

For each value of $k = 1, 2, 3, 4, 5$, find necessary and sufficient conditions on the number $a > 0$ so that there exists a tetrahedron with $k$ edges of length $a$, and the remaining $6 - k$ edges of length 1.

## 1969/4.

A semicircular arc $\gamma$ is drawn on $AB$ as diameter. $C$ is a point on $\gamma$ other than $A$ and $B$, and $D$ is the foot of the perpendicular from $C$ to $AB$. We consider three circles, $\gamma_1, \gamma_2, \gamma_3$, all tangent to the line $AB$. Of these, $\gamma_1$ is inscribed in $\Delta ABC$, while $\gamma_2$ and $\gamma_3$ are both tangent to $CD$ and to $\gamma$, one on each side of $CD$. Prove that $\gamma_1, \gamma_2$ and $\gamma_3$ have a second tangent in common.

## 1969/5.

Given $n > 4$ points in the plane such that no three are collinear. Prove that there are at least $\binom{n-3}{2}$ convex quadrilaterals whose vertices are four of the given points.

## 1969/6.

Prove that for all real numbers $x_1, x_2, y_1, y_2, z_1, z_2$, with $x_1 > 0, x_2 > 0, x_1y_1 - z_1^2 > 0, x_2y_2 - z_2^2 > 0$, the inequality

$$\frac{8}{(x_1 + x_2)(y_1 + y_2) - (z_1 + z_2)^2} \leq \frac{1}{x_1y_1 - z_1^2} + \frac{1}{x_2y_2 - z_2^2}$$

is satisfied. Give necessary and sufficient conditions for equality.

# Twelfth International Olympiad, 1970

## 1970/1.

Let $M$ be a point on the side $AB$ of $\Delta ABC$. Let $r_1, r_2$ and $r$ be the radii of the inscribed circles of triangles $AMC, BMC$ and $ABC$. Let $q_1, q_2$ and $q$ be the radii of the escribed circles of the same triangles that lie in the angle $ACB$. Prove that

$$\frac{r_1}{q_1} \cdot \frac{r_2}{q_2} = \frac{r}{q}.$$

## 1970/2.

Let $a, b$ and $n$ be integers greater than 1, and let $a$ and $b$ be the bases of two number systems. $A_{n-1}$ and $A_n$ are numbers in the system with base $a$, and $B_{n-1}$ and $B_n$are numbers in the system with base $b$; these are related as follows:

$$\begin{aligned} A_n &= x_n x_{n-1} \cdots x_0, A_{n-1} = x_{n-1} x_{n-2} \cdots x_0, \\ B_n &= x_n x_{n-1} \cdots x_0, B_{n-1} = x_{n-1} x_{n-2} \cdots x_0, \\ x_n &\neq 0, x_{n-1} \neq 0. \end{aligned}$$

Prove:

$$\frac{A_{n-1}}{A_n} < \frac{B_{n-1}}{B_n} \text{ if and only if } a > b.$$

## 1970/3.

The real numbers $a_0, a_1, ..., a_n, ...$ satisfy the condition:

$$1 = a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_n \leq \cdots.$$

The numbers $b_1, b_2, ..., b_n, ...$ are defined by

$$b_n = \sum_{k=1}^{n} \left(1 - \frac{a_{k-1}}{a_k}\right) \frac{1}{\sqrt{a_k}}.$$

(a) Prove that $0 \leq b_n < 2$ for all $n$.
(b) Given $c$ with $0 \leq c < 2$, prove that there exist numbers $a_0, a_1, ...$ with the above properties such that $b_n > c$ for large enough $n$.

## 1970/4.

Find the set of all positive integers $n$ with the property that the set $\{n, n+1, n+2, n+3, n+4, n+5\}$ can be partitioned into two sets such that the product of the numbers in one set equals the product of the numbers in the other set.

## 1970/5.

In the tetrahedron $ABCD$, angle $BDC$ is a right angle. Suppose that the foot $H$ of the perpendicular from $D$ to the plane $ABC$ is the intersection of the altitudes of $\Delta ABC$. Prove that

$$(AB + BC + CA)^2 \leq 6(AD^2 + BD^2 + CD^2).$$

For what tetrahedra does equality hold?

## 1970/6.

In a plane there are 100 points, no three of which are collinear. Consider all possible triangles having these points as vertices. Prove that no more than 70% of these triangles are acute-angled.

# Thirteenth International Olympiad, 1971

## 1971/1.

Prove that the following assertion is true for $n = 3$ and $n = 5$, and that it is false for every other natural number $n > 2$ :
If $a_1, a_2, ..., a_n$ are arbitrary real numbers, then
$(a_1 - a_2)(a_1 - a_3) \cdots (a_1 - a_n) + (a_2 - a_1)(a_2 - a_3) \cdots (a_2 - a_n)$
$+ \cdots + (a_n - a_1)(a_n - a_2) \cdots (a_n - a_{n-1}) \geq 0$

## 1971/2.

Consider a convex polyhedron $P_1$ with nine vertices $A_1A_2, ..., A_9$; let $P_i$ be the polyhedron obtained from $P_1$ by a translation that moves vertex $A_1$ to $A_i (i = 2, 3, ..., 9)$. Prove that at least two of the polyhedra $P_1, P_2, ..., P_9$ have an interior point in common.

## 1971/3.

Prove that the set of integers of the form $2^k - 3 (k = 2, 3, ...)$ contains an infinite subset in which every two members are relatively prime.

## 1971/4.

All the faces of tetrahedron $ABCD$ are acute-angled triangles. We consider all closed polygonal paths of the form $XYZTX$ defined as follows: $X$ is a point on edge $AB$ distinct from $A$ and $B$; similarly, $Y, Z, T$ are interior points of edges $BCCD, DA$, respectively. Prove:
(a) If $\angle DAB + \angle BCD \neq \angle CDA + \angle ABC$, then among the polygonal paths, there is none of minimal length.
(b) If $\angle DAB + \angle BCD = \angle CDA + \angle ABC$, then there are infinitely many shortest polygonal paths, their common length being $2AC \sin(\alpha/2)$, where $\alpha = \angle BAC + \angle CAD + \angle DAB$.

## 1971/5.

Prove that for every natural number $m$, there exists a finite set $S$ of points in a plane with the following property: For every point $A$ in $S$, there are exactly $m$ points in $S$ which are at unit distance from $A$.

## 1971/6.

Let $A = (a_{ij})(i, j = 1, 2, ..., n)$ be a square matrix whose elements are non-negative integers. Suppose that whenever an element $a_{ij} = 0$, the sum of the elements in the $i$th row and the $j$th column is $\geq n$. Prove that the sum of all the elements of the matrix is $\geq n^2/2$.

# Fourteenth International Olympiad, 1972

## 1972/1.

Prove that from a set of ten distinct two-digit numbers (in the decimal system), it is possible to select two disjoint subsets whose members have the same sum.

## 1972/2.

Prove that if $n \geq 4$, every quadrilateral that can be inscribed in a circle can be dissected into $n$ quadrilaterals each of which is inscribable in a circle.

## 1972/3.

Let $m$ and $n$ be arbitrary non-negative integers. Prove that

$$\frac{(2m)!(2n)!}{m!n!(m+n)!}$$

is an integer. $(0! = 1.)$

## 1972/4.

Find all solutions $(x_1, x_2, x_3, x_4, x_5)$ of the system of inequalities

$$\begin{array}{rcl}
(x_1^2 - x_3x_5)(x_2^2 - x_3x_5) & \leq & 0 \\
(x_2^2 - x_4x_1)(x_3^2 - x_4x_1) & \leq & 0 \\
(x_3^2 - x_5x_2)(x_4^2 - x_5x_2) & \leq & 0 \\
(x_4^2 - x_1x_3)(x_5^2 - x_1x_3) & \leq & 0 \\
(x_5^2 - x_2x_4)(x_1^2 - x_2x_4) & \leq & 0
\end{array}$$

where $x_1, x_2, x_3, x_4, x_5$ are positive real numbers.

## 1972/5.

Let $f$ and $g$ be real-valued functions defined for all real values of $x$ and $y$, and satisfying the equation

$$f(x+y) + f(x-y) = 2f(x)g(y)$$

for all $x, y$. Prove that if $f(x)$ is not identically zero, and if $|f(x)| \leq 1$ for all $x$, then $|g(y)| \leq 1$ for all $y$.

## 1972/6.

Given four distinct parallel planes, prove that there exists a regular tetrahedron with a vertex on each plane.

# Fifteenth International Olympiad, 1973

## 1973/1.

Point $O$ lies on line $g; \overrightarrow{OP_1}, \overrightarrow{OP_2}, ..., \overrightarrow{OP_n}$ are unit vectors such that points $P_1, P_2, ..., P_n$ all lie in a plane containing $g$ and on one side of $g$. Prove that if $n$ is odd,

$$\left|\overrightarrow{OP_1} + \overrightarrow{OP_2} + \cdots + \overrightarrow{OP_n}\right| \geq 1$$

Here $\left|\overrightarrow{OM}\right|$ denotes the length of vector $\overrightarrow{OM}$.

## 1973/2.

Determine whether or not there exists a finite set $M$ of points in space not lying in the same plane such that, for any two points $A$ and $B$ of $M$, one can select two other points $C$ and $D$ of $M$ so that lines $AB$ and $CD$ are parallel and not coincident.

## 1973/3.

Let $a$ and $b$ be real numbers for which the equation

$$x^4 + ax^3 + bx^2 + ax + 1 = 0$$

has at least one real solution. For all such pairs $(a, b)$, find the minimum value of $a^2 + b^2$.

## 1973/4.

A soldier needs to check on the presence of mines in a region having the shape of an equilateral triangle. The radius of action of his detector is equal to half the altitude of the triangle. The soldier leaves from one vertex of the triangle. What path shouid he follow in order to travel the least possible distance and still accomplish his mission?

## 1973/5.

$G$ is a set of non-constant functions of the real variable $x$ of the form

$$f(x) = ax + b, a \text{ and } b \text{ are real numbers,}$$

and $G$ has the following properties:
(a) If $f$ and $g$ are in $G$, then $g \circ f$ is in $G$; here $(g \circ f)(x) = g[f(x)]$.
(b) If $f$ is in $G$, then its inverse $f^{-1}$ is in $G$; here the inverse of $f(x) = ax + b$ is $f^{-1}(x) = (x - b)/a$.
(c) For every $f$ in $G$, there exists a real number $x_f$ such that $f(x_f) = x_f$.
Prove that there exists a real number $k$ such that $f(k) = k$ for all $f$ in $G$.

**1973/6.**

Let $a_1, a_2, ..., a_n$ be $n$ positive numbers, and let $q$ be a given real number such that $0 < q < 1$. Find $n$ numbers $b_1, b_2, ..., b_n$ for which

(a) $a_k < b_k$ for $k = 1, 2, \cdots, n$,

(b) $q < \frac{b_{k+1}}{b_k} < \frac{1}{q}$ for $k = 1, 2, ..., n-1$,

(c) $b_1 + b_2 + \cdots + b_n < \frac{1+q}{1-q}(a_1 + a_2 + \cdots + a_n)$.

# Sixteenth International Olympiad, 1974

## 1974/1.

Three players $A, B$ and $C$ play the following game: On each of three cards an integer is written. These three numbers $p, q, r$ satisfy $0 < p < q < r$. The three cards are shuffled and one is dealt to each player. Each then receives the number of counters indicated by the card he holds. Then the cards are shuffled again; the counters remain with the players.
This process (shuffling, dealing, giving out counters) takes place for at least two rounds. After the last round, $A$ has 20 counters in all, $B$ has 10 and $C$ has 9. At the last round $B$ received $r$ counters. Who received $q$ counters on the first round?

## 1974/2.

In the triangle $ABC$, prove that there is a point $D$ on side $AB$ such that $CD$ is the geometric mean of $AD$ and $DB$ if and only if

$$\sin A \sin B \leq \sin^2 \frac{C}{2}.$$

## 1974/3.

Prove that the number $\sum_{k=0}^{n} \binom{2n+1}{2k+1} 2^{3k}$ is not divisible by 5 for any integer $n \geq 0$.

## 1974/4.

Consider decompositions of an $8 \times 8$ chessboard into $p$ non-overlapping rectangles subject to the following conditions:
(i) Each rectangle has as many white squares as black squares.
(ii) If $a_i$ is the number of white squares in the $i$-th rectangle, then $a_1 < a_2 < \cdots < a_p$. Find the maximum value of $p$ for which such a decomposition is possible. For this value of $p$, determine all possible sequences $a_1, a_2, \cdots, a_p$.

## 1974/5.

Determine all possible values of

$$S = \frac{a}{a+b+d} + \frac{b}{a+b+c} + \frac{c}{b+c+d} + \frac{d}{a+c+d}$$

where $a, b, c, d$ are arbitrary positive numbers.

**1974/6.**

Let $P$ be a non-constant polynomial with integer coefficients. If $n(P)$ is the number of distinct integers $k$ such that $(P(k))^2 = 1$, prove that $n(P) - \deg(P) \leq 2$, where $\deg(P)$ denotes the degree of the polynomial $P$.

## Seventeenth International Olympiad, 1975

### 1975/1.

Let $x_i, y_i$ $(i = 1, 2, ..., n)$ be real numbers such that

$$x_1 \geq x_2 \geq \cdots \geq x_n \text{ and } y_1 \geq y_2 \geq \cdots \geq y_n.$$

Prove that, if $z_1, z_2, \cdots, z_n$ is any permutation of $y_1, y_2, \cdots, y_n$, then

$$\sum_{i=1}^{n}(x_i - y_i)^2 \leq \sum_{i=1}^{n}(x_i - z_i)^2.$$

### 1975/2.

Let $a_1, a_2, a_{3,} \cdots$ be an infinite increasing sequence of positive integers. Prove that for every $p \geq 1$ there are infinitely many $a_m$ which can be written in the form

$$a_m = xa_p + ya_q$$

with $x, y$ positive integers and $q > p$.

### 1975/3.

On the sides of an arbitrary triangle $ABC$, triangles $ABR, BCP, CAQ$ are constructed externally with $\angle CBP = \angle CAQ = 45°, \angle BCP = \angle ACQ = 30°, \angle ABR = \angle BAR = 15°$. Prove that $\angle QRP = 90°$ and $QR = RP$.

### 1975/4.

When $4444^{4444}$ is written in decimal notation, the sum of its digits is $A$. Let $B$ be the sum of the digits of $A$. Find the sum of the digits of $B$. ($A$ and $B$ are written in decimal notation.)

### 1975/5.

Determine, with proof, whether or not one can find 1975 points on the circumference of a circle with unit radius such that the distance between any two of them is a rational number.

### 1975/6.

Find all polynomials $P$, in two variables, with the following properties:
(i) for a positive integer $n$ and all real $t, x, y$

$$P(tx, ty) = t^n P(x, y)$$

(that is, $P$ is homogeneous of degree $n$),
(ii) for all real $a, b, c$,

$$P(b+c, a) + P(c+a, b) + P(a+b, c) = 0,$$

(iii) $P(1, 0) = 1$.

# Eighteenth International Olympiad, 1976

## 1976/1.

In a plane convex quadrilateral of area 32, the sum of the lengths of two opposite sides and one diagonal is 16. Determine all possible lengths of the other diagonal.

## 1976/2.

Let $P_1(x) = x^2 - 2$ and $P_j(x) = P_1(P_{j-1}(x))$ for $j = 2, 3, \cdots$. Show that, for any positive integer $n$, the roots of the equation $P_n(x) = x$ are real and distinct.

## 1976/3.

A rectangular box can be filled completely with unit cubes. If one places as many cubes as possible, each with volume 2, in the box, so that their edges are parallel to the edges of the box, one can fill exactly 40% of the box. Determine the possible dimensions of all such boxes.

## 1976/4.

Determine, with proof, the largest number which is the product of positive integers whose sum is 1976.

## 1976/5.

Consider the system of $p$ equations in $q = 2p$ unknowns $x_1, x_2, \cdots, x_q$ :

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1q}x_q &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2q}x_q &= 0 \\ &\cdots \\ a_{p1}x_1 + a_{p2}x_2 + \cdots + a_{pq}x_q &= 0 \end{aligned}$$

with every coefficient $a_{ij}$ member of the set $\{-1, 0, 1\}$. Prove that the system has a solution $(x_1, x_2, \cdots, x_q)$ such that
(a) all $x_j$ $(j = 1, 2, ..., q)$ are integers,
(b) there is at least one value of $j$ for which $x_j \neq 0$,
(c) $|x_j| \leq q (j = 1, 2, ..., q)$.

## 1976/6.

A sequence $\{u_n\}$ is defined by

$$u_0 = 2, u_1 = 5/2, u_{n+1} = u_n(u_{n-1}^2 - 2) - u_1 \text{for } n = 1, 2, \cdots$$

Prove that for positive integers $n$,

$$[u_n] = 2^{[2^n - (-1)^n]/3}$$

where $[x]$ denotes the greatest integer $\leq x$.

# Nineteenth International Mathematical Olympiad, 1977

## 1977/1.

Equilateral triangles $ABK, BCL, CDM, DAN$ are constructed inside the square $ABCD$. Prove that the midpoints of the four segments $KL, LM, MN, NK$ and the midpoints of the eight segments $AKBK, BL, CL, CM, DM, DN, AN$ are the twelve vertices of a regular dodecagon.

## 1977/2.

In a finite sequence of real numbers the sum of any seven successive terms is negative, and the sum of any eleven successive terms is positive. Determine the maximum number of terms in the sequence.

## 1977/3.

Let $n$ be a given integer $> 2$, and let $V_n$ be the set of integers $1 + kn$, where $k = 1, 2, ....$ A number $m \in V_n$ is called *indecomposable* in $V_n$ if there do not exist numbers $p, q \in V_n$ such that $pq = m$. Prove that there exists a number $r \in V_n$ that can be expressed as the product of elements indecomposable in $V_n$ in more than one way. (Products which differ only in the order of their factors will be considered the same.)

## 1977/4.

Four real constants $a, b, A, B$ are given, and

$$f(\theta) = 1 - a\cos\theta - b\sin\theta - A\cos 2\theta - B\sin 2\theta.$$

Prove that if $f(\theta) \geq 0$ for all real $\theta$, then

$$a^2 + b^2 \leq 2 \text{ and } A^2 + B^2 \leq 1.$$

## 1977/5.

Let $a$ and $b$ be positive integers. When $a^2+b^2$ is divided by $a+b$, the quotient is $q$ and the remainder is $r$. Find all pairs $(a, b)$ such that $q^2 + r = 1977$.

## 1977/6.

Let $f(n)$ be a function defined on the set of all positive integers and having all its values in the same set. Prove that if

$$f(n + 1) > f(f(n))$$

for each positive integer $n$, then

$$f(n) = n \text{ for each } n.$$