La longitud de una circunferencia de radio r viene dada por:

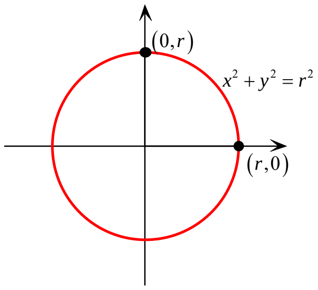
$$L = 2\pi r$$

Demostración:

La longitud de una curva $y = f(x)$ entre dos valores a y b viene dada por:

$$L = \int_a^b \sqrt{1 + \left[f'(x)\right]^2}\,dx$$

En nuestro caso, se tiene:



$$y^2 = r^2 - x^2 \Rightarrow y = \sqrt{r^2 - x^2}$$

$$f(x) = \sqrt{r^2 - x^2}$$

$$f'(x) = \frac{-2x}{2\sqrt{r^2 - x^2}} = \frac{-x}{\sqrt{r^2 - x^2}}$$

y, por tanto,

$$L = 4\int_0^r \sqrt{1 + \frac{x^2}{r^2 - x^2}}\,dx = 4\int_0^r \sqrt{\frac{r^2}{r^2 - x^2}}\,dx = 4\int_0^r \frac{r}{\sqrt{r^2 - x^2}}\,dx = \left[\begin{array}{lcl} \operatorname{sen}\theta = \frac{r}{x} & \Rightarrow & x = r\operatorname{sen}\theta \\ & & dx = r\cos\theta\, d\theta \\ x = 0 & \Rightarrow & \theta = 0 \\ x = r & \Rightarrow & \theta = \frac{\pi}{2} \end{array}\right] =$$

$$= 4\int_0^{\pi/2} \frac{r^2\cos\theta\, d\theta}{\sqrt{r^2\left(1 - \operatorname{sen}^2\theta\right)}} = 4\int_0^{\pi/2} \frac{r^2\cos\theta\, d\theta}{\sqrt{r^2}\sqrt{1 - \operatorname{sen}^2\theta}} = 4\int_0^{\pi/2} \frac{r^2\cos\theta\, d\theta}{r\cos\theta} = 4r\int_0^{\pi/2} d\theta = 4r\frac{\pi}{2} = 2\pi r$$

C.Q.D.

El área de un círculo de radio r viene dada por:

$$A = \pi r^2$$

Demostración:

$$A = 2\int_{-r}^{r} \sqrt{r^2 - x^2}\,dx = 4\int_0^r \sqrt{r^2 - x^2}\,dx = \left[\begin{array}{lcl} x = \operatorname{sen} t & \Rightarrow & dx = \cos t\,dt \\ x = 0 & \Rightarrow & t = 0 \\ x = r & \Rightarrow & t = \frac{\pi}{2} \end{array}\right] = 4\int_0^{\frac{\pi}{2}} \sqrt{r^2 - r^2\operatorname{sen}^2\theta} \cdot r\cos\theta\, d\theta =$$

$$= 4\int_0^{\frac{\pi}{2}} r^2\cos^2\theta\, d\theta = 4r^2\int_0^{\frac{\pi}{2}} \frac{1 + \cos 2\theta}{2}\,d\theta = 4r^2\left(\frac{1}{2}\theta + \frac{\operatorname{sen} 2\theta}{4}\right)\Bigg|_0^{\frac{\pi}{2}} = 4r^2$$

C.Q.D.