Ejercicios para practicar: trigonometría

1. Resuelve las siguientes ecuaciones trigonométricas:

1) $5\operatorname{sen} x = 2$
2) $7\cos x = -1$
3) $5\operatorname{tg} x = 12$
4) $2\operatorname{tg} x = 2$
5) $\operatorname{sen}(2x) = 1$
6) $\cos x + \cos(2x) = 0$

2. Resuelve las siguientes ecuaciones trigonométricas:

1) $2\cos^2 x + \cos x - 1 = 0$
2) $2\operatorname{sen}^2 x - 1 = 0$
3) $\operatorname{tg}^2 x - \operatorname{tg} x = 0$
4) $2\operatorname{sen}^2 x + 3\cos x = 3$
5) $4\cos(2x) + 3\cos x = 1$
6) $\operatorname{tg}(2x) + 2\cos x = 0$
7) $\sqrt{2}\cos\frac{x}{2} - \cos x = 1$
8) $2\operatorname{sen} x\cos^2 x - 6\operatorname{sen}^3 x = 0$

3. Demuestra las siguientes igualdades:

1) $\dfrac{1-\cos^2 x}{\operatorname{sen}(2x)} = \dfrac{\operatorname{tg} x}{2}$
2) $\operatorname{sen}(2x) = \dfrac{2\operatorname{tg} x}{1+\operatorname{tg}^2 x}$
3) $\dfrac{2\operatorname{sen} x - \operatorname{sen}(2x)}{2\operatorname{sen} x + \operatorname{sen}(2x)} = \dfrac{1-\cos x}{1+\cos x}$
4) $\left(\dfrac{\operatorname{tg}\frac{x}{2}}{\operatorname{sen} x} - 2\operatorname{sen}^2\left(\dfrac{x}{2}\right)\right)\cos^2\left(\dfrac{x}{2}\right) = \cos^2\left(\dfrac{x}{2}\right)$

4. Demuestra la siguiente identidad:

$$\operatorname{sen} x\operatorname{sen}(x-y) + \cos x\cos(x-y) = \cos y$$

Chuletario de fórmulas que hay que saberse:

$\operatorname{sen}(90^\text{o}-\alpha) = \cos\alpha$ $\cos(90^\text{o}-\alpha) = \operatorname{sen}\alpha$	$\operatorname{sen}(180^\text{o}-\alpha) = \operatorname{sen}\alpha$ $\cos(180^\text{o}-\alpha) = -\cos\alpha$	$\operatorname{sen}(180^\text{o}+\alpha) = -\operatorname{sen}\alpha$ $\cos(180^\text{o}+\alpha) = -\cos\alpha$
$\operatorname{sen}(-\alpha) = \operatorname{sen}(360^\text{o}-\alpha) = -\operatorname{sen}\alpha$ $\cos(-\alpha) = \cos(360^\text{o}-\alpha) = \cos\alpha$		
$\operatorname{sen}^2\alpha + \cos^2\alpha = 1 \Rightarrow \begin{cases} 1+\operatorname{cotg}^2\alpha = \operatorname{cosec}^2\alpha \\ 1+\operatorname{tg}^2\alpha = \sec^2\alpha \end{cases}$		
$\operatorname{sen}(\alpha\pm\beta) = \operatorname{sen}\alpha\cos\beta \pm \cos\alpha\operatorname{sen}\beta$ $\cos(\alpha\pm\beta) = \cos\alpha\cos\beta \mp \operatorname{sen}\alpha\operatorname{sen}\beta$ $\operatorname{tg}(\alpha\pm\beta) = \dfrac{\operatorname{tg}\alpha \pm \operatorname{tg}\beta}{1\mp\operatorname{tg}\alpha\operatorname{tg}\beta}$		
$\operatorname{sen}(2\alpha) = 2\operatorname{sen}\alpha\cos\alpha$ $\cos(2\alpha) = \cos^2\alpha - \operatorname{sen}^2\alpha$ $\operatorname{tg}(2\alpha) = \dfrac{2\operatorname{tg}\alpha}{1-\operatorname{tg}^2\alpha}$		
$\operatorname{sen}\left(\dfrac{\alpha}{2}\right) = \pm\sqrt{\dfrac{1-\cos\alpha}{2}}$ $\cos\left(\dfrac{\alpha}{2}\right) = \pm\sqrt{\dfrac{1+\cos\alpha}{2}}$ $\operatorname{tg}\left(\dfrac{\alpha}{2}\right) = \pm\sqrt{\dfrac{1-\cos\alpha}{1+\cos\alpha}}$		

SOLUCIONES

SOLUCIÓN EJERCICIO 1:

1) $5\,\text{sen}\,x = 2 \Rightarrow \text{sen}\,x = \dfrac{2}{5} \Rightarrow \begin{cases} x_1 = 23^\text{o}\,34'41,44'' \\ x_2 = 180^\text{o} - 23^\text{o}\,34'41,44'' = 156^\text{o}\,25'18,56'' \end{cases}$

2) $7\cos x = -1 \Rightarrow \cos x = -\dfrac{1}{7} \Rightarrow \begin{cases} x_1 = \arccos\left(-\dfrac{1}{7}\right) = 98^\text{o}\,12'47,56'' \\ x_2 = 180^\text{o} + 98^\text{o}\,12'47,56' = 261^\text{o}\,47'12,44'' \end{cases}$

3) $5\,\text{tg}\,x = 12 \Rightarrow \text{tg}\,x = \dfrac{12}{5} \Rightarrow \begin{cases} x_1 = \text{arctg}\,\dfrac{12}{5} = 67^\text{o}\,22'48,49'' \\ x_2 = 247^\text{o}\,22'48,49'' \end{cases}$

4) $2\,\text{tg}\,x = 2 \Rightarrow \text{tg}\,x = 1 \Rightarrow \begin{cases} x_1 = \text{arctg}\,1 = 45^\text{o} \\ x_2 = 225^\text{o} \end{cases}$

5) $\text{sen}(2x) = 1 \Rightarrow 2x = 90^\text{o} \Rightarrow x = 45^\text{o}$

6) $\cos x + \cos(2x) = 0 \Rightarrow \cos x + 2\,\text{sen}\,x\cos x = 0 \Rightarrow \cos x(1 + 2\,\text{sen}\,x) = 0 \Rightarrow$

$$\Rightarrow \begin{cases} \cos x = 0 \Rightarrow x_1 = \arccos 0 = 90^\text{o} \text{ y } x_2 = 270^\text{o} \\ 1 + 2\,\text{sen}\,x = 0 \Rightarrow \text{sen}\,x = -\dfrac{1}{2} \Rightarrow x_3 = \text{arcsen}\left(-\dfrac{1}{2}\right) = 210^\text{o} \text{ y } x_4 = 330^\text{o} \end{cases}$$

SOLUCIÓN EJERCICIO 2:

1) Es una ecuación de segundo grado en $\cos x$:

$$\cos x = \frac{-1 \pm \sqrt{1 - 4\cdot 2\cdot(-1)}}{4} = \begin{cases} \dfrac{1}{2} \Rightarrow \cos x = \dfrac{1}{2} \Rightarrow x_1 = 60^\text{o} \text{ y } x_2 = 300^\text{o} \\ -1 \Rightarrow \cos x = -1 \Rightarrow x_3 = 180^\text{o} \end{cases}$$

Las tres soluciones obtenidas son válidas.

2) $2\,\text{sen}^2 x - 1 = 0 \Rightarrow \text{sen}^2 x = \dfrac{1}{2} \Rightarrow \text{sen}\,x = \pm\sqrt{\dfrac{1}{2}} = \pm\dfrac{\sqrt{2}}{2}$

Si $\text{sen}\,x = \dfrac{\sqrt{2}}{2} \Rightarrow x_1 = 45^\text{o}$ y $x_2 = 180^\text{o} - 45^\text{o} = 135^\text{o}$

Si $\text{sen}\,x = -\dfrac{\sqrt{2}}{2} \Rightarrow x_3 = -45^\text{o} = 315^\text{o}$ y $x_4 = 180^\text{o} + 45^\text{o} = 225^\text{o}$

Las cuatro soluciones son válidas.

3) $\text{tg}^2 x - \text{tg}\,x = 0 \Rightarrow \text{tg}\,x(\text{tg}\,x - 1) = 0 \Rightarrow \begin{cases} \text{tg}\,x = 0 \Rightarrow x_1 = \text{arctg}\,0 = 0^\text{o} \text{ y } x_2 = 180^\text{o} \\ 1 - \text{tg}\,x = 0 \Rightarrow \text{tg}\,x = 1 \Rightarrow x_3 = \text{arctg}\,1 = 45^\text{o} \text{ y } x_4 = 225^\text{o} \end{cases}$

Las cuatro soluciones obtenidas son válidas.

4) $2\,\text{sen}^2 x + 3\cos x = 3 \Rightarrow \left[\text{sen}^2 x + \cos^2 x = 1\right]\ 2\left(1-\cos^2 x\right) + 3\cos x = 3 \Rightarrow -2\cos^2 x + 3\cos x + 1 = 0$

que es una ecuación de segundo grado en $\cos x$:

$$\cos x = \frac{3 \pm \sqrt{9-4\cdot 2\cdot 1}}{2\cdot 2} = \begin{cases} 1 \Rightarrow \cos x = 1 \Rightarrow x_1 = \arccos 1 = 0^\text{o} \\ \dfrac{1}{2} \Rightarrow \cos x = \dfrac{1}{2} \Rightarrow x_2 = \arccos\dfrac{1}{2} = 60^\text{o} \ \text{ y } x_3 = -60^\text{o} = 300^\text{o} \end{cases}$$

Las tres son soluciones de la ecuación original.

5) $4\cos(2x) + 3\cos x = 1 \Rightarrow 4\left(\cos^2 x - \text{sen}^2 x\right) + 3\cos x = 1 \Rightarrow \left[\text{sen}^2 x = 1 - \cos^2 x\right]$

$4\left[\cos^2 x - \left(1-\cos^2 x\right)\right] + 3\cos x = 1 \Rightarrow 4\left[2\cos^2 x - 1\right] + 3\cos x = 1 \Rightarrow 8\cos^2 x - 4 + 3\cos x = 1$

$\Rightarrow 8\cos^2 x + 3\cos x - 5 = 0$, que es una ecuación de segundo grado en $\cos x$:

$$\cos x = \frac{-3 \pm 13}{16} = \begin{cases} \dfrac{5}{8} \Rightarrow \cos x = \dfrac{5}{8} \Rightarrow x_1 = \arccos\dfrac{5}{8} = 51^\text{o}19'4{,}13" \ \text{ y } x_2 = -51^\text{o}19'4{,}13" \\ -1 \Rightarrow \cos x = -1 \Rightarrow x_3 = \arccos(-1) = 180^\text{o} \end{cases}$$

Las tres soluciones son válidas.

6) $\text{tg}(2x) + 2\cos x = 0 \Rightarrow \dfrac{2\,\text{tg}\, x}{1-\text{tg}^2 x} + 2\cos x = 0 \Rightarrow$ (el 2 se puede simplificar)

$$\frac{\frac{\text{sen}\, x}{\cos x}}{1 - \frac{\text{sen}^2 x}{\cos^2 x}} + \cos x = 0 \Rightarrow \frac{\frac{\text{sen}\, x}{\cos x}}{\frac{\cos^2 x - \text{sen}^2 x}{\cos^2 x}} + \cos x = 0 \Rightarrow \frac{\text{sen}\, x \cos x}{\cos^2 x - \text{sen}^2 x} + \cos x = 0 \Rightarrow$$

$$\Rightarrow \frac{\text{sen}\, x\cos x}{\cos^2 x - \text{sen}^2 x} + \frac{\cos x\left(\cos^2 x - \text{sen}^2 x\right)}{\cos^2 x - \text{sen}^2 x} = 0 \Rightarrow \text{sen}\, x \cos x + \cos x\left(\cos^2 x - \text{sen}^2 x\right) = 0$$

$\Rightarrow$ (sacando factor común) $\cos x\left(\text{sen}\, x + \cos^2 x - \text{sen}^2 x\right) = 0 \Rightarrow \left[\cos^2 x = 1 - \text{sen}^2 x\right]$

$$\Rightarrow \cos x\left(\text{sen}\, x + 1 - \text{sen}^2 x - \text{sen}^2 x\right) = 0 \Rightarrow \begin{cases} \cos x = 0 \\ 1 + \text{sen}\, x - 2\,\text{sen}^2 x = 0 \Rightarrow \text{sen}\, x = \dfrac{-1 \pm 3}{-4} = \begin{cases} -\dfrac{1}{2} \\ 1 \end{cases} \end{cases}$$

Si $\cos x = 0 \Rightarrow x_1 = 90^\text{o}$ y $x_2 = 270^\text{o}$

$\text{sen}\, x = -\dfrac{1}{2} \Rightarrow x_3 = 210^\text{o}$ y $x_4 = 330^\text{o} = -30^\text{o}$

$\text{sen}\, x = 1 \Rightarrow x_5 = 90^\text{o} = x_1$

Las cuatro son soluciones válidas.

7) $\sqrt{2}\cos\dfrac{x}{2} - \cos x = 1 \Rightarrow \sqrt{2}\sqrt{\dfrac{1+\cos x}{2}} - \cos x = 1 \Rightarrow \sqrt{1+\cos x} - \cos x = 1 \Rightarrow$

$\sqrt{1+\cos x} = 1 + \cos x \Rightarrow$ (elevando al cuadrado) $1 + \cos x = 1 + \cos^2 x + 2\cos x \Rightarrow$

$$\Rightarrow \cos^2 x + \cos x = 0 \Rightarrow \cos x(\cos x + 1) = 0 \Rightarrow \begin{cases} \cos x = 0 \Rightarrow x_1 = 90^\text{o} \ \text{ y } x_2 = 270^\text{o} \\ \cos x = -1 \Rightarrow x_3 = 180^\text{o} \end{cases}$$

Al comprobar, se obtiene que las únicas soluciones válidas son 90º y 180º.

8) $2\operatorname{sen} x \cos^2 x - 6\operatorname{sen}^3 x = 0 \Rightarrow 2\operatorname{sen} x\left(\cos^2 x - 3\operatorname{sen}^2 x\right) = 0 \Rightarrow \left[\cos^2 x = 1 - \operatorname{sen}^2 x\right]$

$$2\operatorname{sen} x\left(\cos^2 x + \operatorname{sen}^2 x - 4\operatorname{sen}^2 x\right) = 0 \Rightarrow 2\operatorname{sen} x\left(1 - 4\operatorname{sen}^2 x\right) = 0 \Rightarrow \begin{cases} \operatorname{sen} x = 0 \\ \operatorname{sen}^2 x = \dfrac{1}{4} \end{cases}$$

Si $\operatorname{sen} x = 0 \Rightarrow x_1 = 0^\circ$ y $x_2 = 180^\circ$

$$\operatorname{sen}^2 x = \frac{1}{4} \Rightarrow \operatorname{sen} x = \pm\sqrt{\frac{1}{4}} = \pm\frac{1}{2} \Rightarrow x_3 = 30^\circ,\ x_4 = 150^\circ,\ x_5 = 210^\circ \text{ y } x_6 = 330^\circ$$

Las seis soluciones obtenidas son válidas.

SOLUCIÓN EJERCICIO 3:

1) $\dfrac{1-\cos^2 x}{\operatorname{sen}(2x)} = \dfrac{\operatorname{tg} x}{2}$

$$\frac{1-\cos^2 x}{\operatorname{sen}(2x)} = \frac{\operatorname{sen}^2 x}{2\operatorname{sen} x\cos x} = \frac{\operatorname{sen} x}{2\cos x} = \frac{\operatorname{tg} x}{2}$$

2) $\operatorname{sen}(2x) = \dfrac{2\operatorname{tg} x}{1+\operatorname{tg}^2 x}$

$$\operatorname{sen}(2x) = 2\operatorname{sen} x\cos x = \frac{2\operatorname{sen} x\cos^2 x}{\cos x} = 2\operatorname{tg} x\cos^2 x = \frac{2\operatorname{tg} x}{\dfrac{1}{\cos^2 x}} = \frac{2\operatorname{tg} x}{1+\operatorname{tg}^2 x}$$

3) $\dfrac{2\operatorname{sen} x - \operatorname{sen}(2x)}{2\operatorname{sen} x + \operatorname{sen}(2x)} = \dfrac{2\operatorname{sen} x - 2\operatorname{sen} x\cos x}{2\operatorname{sen} x + 2\operatorname{sen} x\cos x} = \dfrac{2\operatorname{sen} x(1-\cos x)}{2\operatorname{sen} x(1+\cos x)} = \dfrac{1-\cos x}{1+\operatorname{sen} x}$

4) $\left(\dfrac{\operatorname{tg}\dfrac{x}{2}}{\operatorname{sen} x} - 2\operatorname{sen}^2\left(\dfrac{x}{2}\right)\right)\cos^2\left(\dfrac{x}{2}\right) = \left(\dfrac{\sqrt{\dfrac{1-\cos x}{1+\cos x}}}{\operatorname{sen} x} - (1-\cos x)\right)\dfrac{1+\cos x}{2} =$

$$= \left(\frac{\sqrt{\dfrac{1-\cos x}{1+\cos x}}}{\sqrt{1-\cos^2 x}} - (1-\cos x)\right)\frac{1+\cos x}{2} = \left(\sqrt{\frac{\dfrac{1-\cos x}{1+\cos x}}{1-\cos^2 x}} - (1-\cos x)\right)\frac{1+\cos x}{2} =$$

$$= \left(\sqrt{\frac{(1-\cos x)}{(1+\cos x)(1-\cos^2 x)}} - (1-\cos x)\right)\frac{1+\cos x}{2} =$$

$$= \left(\sqrt{\frac{\cancel{(1-\cos x)}}{(1+\cos x)(1+\cos x)\cancel{(1-\cos x)}}} - (1-\cos x)\right)\frac{1+\cos x}{2} =$$

$$= \left(\sqrt{\frac{1}{(1+\cos x)^2}} - (1-\cos x)\right)\frac{1+\cos x}{2} = \left[\frac{1}{1+\cos x} - (1-\cos x)\right]\frac{1+\cos x}{2} =$$

$$=\left(\frac{1-(1-\cos x)(1+\cos x)}{1+\cos x}\right)\frac{1+\cos x}{2}=\left(\frac{1-(1-\cos^2 x)}{1+\cos x}\right)\frac{1+\cos x}{2}=\left(\frac{\cos^2 x}{\cancel{1+\cos x}}\right)\frac{\cancel{1+\cos x}}{2}=$$

$$=\frac{\cos^2 x}{2}$$

SOLUCIÓN EJERCICIO 4:

$$\operatorname{sen} x \operatorname{sen}(x-y)+\cos x\cos(x-y)=\operatorname{sen} x(\operatorname{sen} x\cos y-\cos x\operatorname{sen} y)+\cos x(\cos x\cos y+\operatorname{sen} x\operatorname{sen} y)=$$

$$\operatorname{sen}^2 x\cos y-\operatorname{sen} x\cos x\operatorname{sen} y+\cos^2 x\cos y+\operatorname{sen} x\cos x\operatorname{sen} y=\cos y\left(\operatorname{sen}^2 x+\cos^2 x\right)=\cos y$$